Exchangeable Occupancy Models and Discrete Processes with the Generalized Uniform Order Statistics Property

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December 6, 2011

Abstract

This work focuses on Exchangeable Occupancy Models (EOM) and their relations with the Uniform Order Statistics Property (UOSP) for point processes in discrete time. As our main purpose, we show how definitions and results presented in Shaked, Spizzichino and Suter (2004) can be unified and generalized in the frame of occupancy models. We first show some general facts about EOM's. Then we introduce a class of EOM's, called $\mathcal{M}^{(a)}$-models, that is used for generalizing the notion of Uniform Order Statistics Property. For processes with this property, we prove a general characterization result. Our interest is also focused on properties of closure w.r.t. some natural transformations of EOM's.

Keywords: Transformations of Occupancy Models, $\mathcal{M}_{n,r}^{(a)}$-Models, Closure Properties, Random Sampling of Arrivals

AMS 2010 Classification: 62G30, 60G09, 60G55

1 Introduction

The so called *occupancy distributions* give rise, as well known, to a class of multivariate models useful in the description of randomized phenomena. The name "occupancy" comes from the interpretation in terms of particles that are randomly distributed among several cells. In particular, three classical examples, related to as many well-known physical systems, belong to this class: Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac model, see Feller (1968). In the Maxwell-Boltzmann (MB) statistics the capacity of each cell is unlimited, and the particles are distinguishable. In the Bose-Einstein (BE) statistics, the capacity of each cell is unlimited but the particles are indistinguishable. In the Fermi Dirac (FD) statistics, the particles are indistinguishable, but cells can only hold a maximum of one particle. All these three statistics

assume that the cells are distinguishable. These models are attractive for many reasons. First of all they have wide range of applications in Sciences, Engineering and also in Statistics, as pointed out from Charalambides (2005, Chapters 4 and 5) and Gadrich and Ravid (2011). Moreover Mahmoud (2009) provided an interpretation of occupancy distributions in terms of Polya Urns, which are very flexible and applicable to problems arising in various areas; e.g. Clinical Trials (see Crimaldi and Leisen (2008) for some references), Economics (see Aruka (2011)) and Computer Science (see Shah, Kothari, Jayadeva and Chandra (2011)). From a probabilistic and combinatoric point of view, the three fundamental models (MB, BE and BS) have many interesting properties. Indeed they are, in particular, *exchangeable* and this is a basic remark for our aims. This paper is in fact concentrated on the theme of Exchangeable Occupancy Models (EOM) and their relations with the Uniform Order Statistics Property (UOSP) of counting processes in discrete time. As one main purpose of ours, we show that some notions and results given in Shaked, Spizzichino and Suter (2004, 2008) admit completely natural generalizations in the frame of EOM's.
We start proving some general properties of the EOM's. In particular, we show that the exchangeability property is maintained under remarkable types of transformations of occupancy models. Then, we introduce the notion of $\mathcal{M}^{(a)}$-models, a relevant sub-class of EOM's that turns out to have an important role in our derivations. In the final part of the paper, we study discrete-time counting processes satisfying the generalized UOSP; in other words, for which the joint distribution of the jump amounts is distributed according to a $\mathcal{M}^{(a)}$-model, conditionally on a fixed number of arrivals up to time t. In particular, for these processes, several characterizations are proved.
More in detail, the outline of the paper is as follows. Our central results will be presented in the Section 5, where we show natural extensions of the results proved in Shaked, Spizzichino and Suter (2004, 2008). Section 2 is devoted to some preliminary arguments about occupancy models and Section 3 presents some specific aspects of the EOM's. In Section 4 we define the class of occupancy $\mathcal{M}^{(a)}$-models and study some related properties.

2 Occupancy models and order statistics of discrete variables

For fixed $n = 2, 3, \dots$ and $r = 1, 2, \dots,$ let $A_{n,r}$ be the set defined by

$$A_{n,r} := \left\{ \mathbf{x} \equiv (x_1, \dots, x_n) : x_j = 0, 1, \dots, r \text{ and } \sum_{j=1}^{n} x_j = r \right\}.$$

As it is well-known (see e.g. Feller (1968)) the cardinality of $A_{n,r}$ is

$$|A_{n,r}| = \binom{n+r-1}{n-1}.$$

Starting from the classical scheme of r *particles* that are distributed stochastically into n *cells*, we consider the random vector $\mathbf{X} \equiv (X_1, X_2, \ldots, X_n)$ where, for $j = 1, 2, \ldots, n$, X_j is the $\{0, 1, \ldots, r\}$-valued random variable that counts the number of particles fallen in j-th cell. $X_1, X_2, \ldots, X_n$ are called *occupancy numbers* and the joint distribution of $(X_1, X_2, \ldots, X_n)$, describing the probabilistic mechanism of assignment of the particles to the cells, is called an *occupancy model*. Since the vector $(X_1, X_2, \ldots, X_n)$ takes its values in the set $A_{n,r}$, an *occupancy model* is then a probability distribution on $A_{n,r}$.

Let now $B_{r,n}$ denote the set

$$B_{r,n} := \{\mathbf{u} \equiv (u_1, u_2, \ldots, u_r) : u_i = 1, 2, \ldots, n \text{ and } u_1 \leq u_2 \leq \cdots \leq u_r\}$$

and consider the mapping $\varphi : B_{r,n} \longrightarrow A_{n,r}$ defined as

$$\varphi(\mathbf{u}) = (\varphi_1(\mathbf{u}), \varphi_2(\mathbf{u}), \ldots, \varphi_n(\mathbf{u}))\,,$$

with

$$\varphi_j(\mathbf{u}) = \sum_{i=1}^{r} \mathbf{1}_{\{u_i = j\}}\,, \qquad \text{for } j = 1, 2, \ldots, n.$$

This is a one-to-one correspondence and then, for the cardinality of $B_{r,n}$, we have

$$|B_{r,n}| = |A_{n,r}| = \binom{n+r-1}{n-1}.$$

As to $\psi = \varphi^{-1} : A_{n,r} \longrightarrow B_{r,n}$, we can write

$$\psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \ldots, \psi_r(\mathbf{x}))\,,$$

with

$$\psi_i(\mathbf{x}) = \min\left\{ s \,\middle|\, \sum_{j=1}^{s} x_j \geq i \right\}\,, \qquad \text{for } i = 1, 2, \ldots, r.$$

We can thus consider the random vector $\mathbf{U} \equiv (U_1, \ldots, U_r)$ defined by

$$\mathbf{U} = \psi\left(\mathbf{X}\right). \tag{1}$$

For $(u_1, \ldots, u_r) \in B_{r,n}$, one then has

$$P\{U_1 = u_1, \ldots, U_r = u_r\} = P\{X_1 = \varphi_1(\mathbf{u}), \ldots, X_n = \varphi_n(\mathbf{u})\}.$$

Let $\mathcal{P}\left(A_{n,r}\right)$ and $\mathcal{P}\left(B_{r,n}\right)$ respectively denote the family of probability distributions on $A_{n,r}$ and the family of probability distributions on $B_{r,n}$. In view of the above one-to-one corre-

spondence between $A_{n,r}$ and $B_{r,n}$, we can consider the induced (one-to-one) correspondence between $\mathcal{P}(A_{n,r})$ and $\mathcal{P}(B_{r,n})$. More precisely, we consider the mappings Φ and $\Psi = \Phi^{-1}$ defined by

$$\Psi(Q)(\mathbf{x}) = Q[\psi(\mathbf{x})] \text{ and } \Phi(P)(\mathbf{u}) = P[\varphi(\mathbf{u})]\,,$$

where $P \in \mathcal{P}(A_{n,r})$ and $Q \in \mathcal{P}(B_{r,n})$.

Remark 2.1. Note that P is the uniform distribution on $A_{n,r}$ (i.e., the Bose-Einstein model) if and only if $Q = \Phi(P)$ is the uniform distribution on $B_{r,n}$.

Consider now r exchangeable random variables $Y_1, Y_2, \ldots, Y_r$, which take values in $\{1, 2, \ldots, n\}$. To these random variables we can associate a vector of occupancy numbers by introducing the random variables $X_1, X_2, \ldots, X_n$ defined by

$$X_j = \sum_{h=1}^{r} \mathbf{1}_{\{Y_h = j\}}\,, \qquad \text{for } j = 1, 2, \ldots, n.$$

The set of the possible values taken by $\mathbf{Y} \equiv (Y_1, \ldots, Y_r)$ is then $D_{r,n} := \{1, 2, ..., n\}^r$.
We will also write $\mathbf{X} = \tilde{\varphi}(\mathbf{Y})$ or

$$X_j = \tilde{\varphi}_j(Y_1, Y_2, \ldots, Y_r), \quad \text{for } j = 1, \ldots, n \tag{2}$$

where $\tilde{\varphi} : D_{r,n} \longrightarrow A_{n,r}$ with

$$\tilde{\varphi}_j(y_1, y_2, \ldots, y_r) = \sum_{h=1}^{r} \mathbf{1}_{\{y_h = j\}}\,, \quad \text{for } \mathbf{y} \in D_{r,n} \text{ and } j = 1, 2, \ldots, n. \tag{3}$$

It can be immediately seen that the probability distribution of $(Y_1, Y_2, \ldots, Y_r)$ is uniquely determined by the probability distribution of $(X_1, X_2, \ldots, X_n)$ and vice versa.
In fact, the following relationships hold

$$P\{Y_1 = y_1, Y_2 = y_2, \ldots, Y_r = y_r\} = \frac{P\{X_1 = \tilde{\varphi}_1(\mathbf{y}), X_2 = \tilde{\varphi}_2(\mathbf{y}), \ldots, X_n = \tilde{\varphi}_n(\mathbf{y})\}}{\binom{r}{\tilde{\varphi}_1(\mathbf{y})\tilde{\varphi}_2(\mathbf{y})\cdots\tilde{\varphi}_n(\mathbf{y})}} \tag{4}$$

and

$$\begin{aligned} &P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} = \\ &\qquad = \binom{r}{x_1 \cdots x_n} P\{Y_1 = \psi_1(\mathbf{x}), Y_2 = \psi_2(\mathbf{x}), \ldots, Y_r = \psi_r(\mathbf{x})\} \end{aligned} \tag{5}$$

We note that $\tilde{\varphi} : D_{r,n} \longrightarrow A_{n,r}$ is different from the previous transformation φ, which is defined on $B_{r,n} \subset D_{r,n}$ and is bijective. However, $\tilde{\varphi}(\mathbf{y}) = \varphi(\mathbf{y})$ if $\mathbf{y} \in B_{r,n}$ and, for any $\mathbf{y} \in D_{r,n}$, we have

$$\tilde{\varphi}(\mathbf{y}) = \tilde{\varphi}(y_{(1)}, \ldots, y_{(r)}) = \varphi(y_{(1)}, \ldots, y_{(r)}),$$

where $(y_{(1)}, \ldots, y_{(r)})$ is the vector of the coordinates of $\mathbf{y}$ rearranged in increasing order.

We consider now the vector $\mathbf{Y}_{(\cdot)} \equiv \left(Y_{(1)}, \ldots, Y_{(r)}\right)$ of the order statistics of the exchangeable vector $(Y_1, Y_2, \ldots, Y_r)$. The set of values taken by $\mathbf{Y}_{(\cdot)}$ is $B_{r,n}$. To a probability distribution of $(Y_1, Y_2, \ldots, Y_r)$ it corresponds one and only one probability distribution of $\mathbf{Y}_{(\cdot)}$ and, for $\mathbf{u} \in B_{r,n}$, we can write

$$P\{Y_1 = u_1, \ldots, Y_r = u_r\} = \frac{P\{Y_{(1)} = u_1, \ldots, Y_{(r)} = u_r\}}{\binom{r}{\varphi_1(\mathbf{u}) \cdots \varphi_n(\mathbf{u})}}.$$

Furthermore,

$$\varphi\left(\mathbf{Y}_{(\cdot)}\right) = \tilde{\varphi}\left(\mathbf{Y}_{(\cdot)}\right) = \tilde{\varphi}\left(\mathbf{Y}\right) = \mathbf{X}\,,$$

and thus

$$\mathbf{Y}_{(\cdot)} = \varphi^{-1}\left(\mathbf{X}\right) = \psi\left(\mathbf{X}\right).$$

We can then summarize the arguments above as follows.

Proposition 2.1. *Let $X_1, X_2, \ldots, X_n$ be random variables such that the support of their joint distribution is $A_{n,r}$, for some $r \in \mathbb{N}$. Moreover, let $Y_1, Y_2, \ldots, Y_r$ be exchangeable, $\{1, 2, \ldots, n\}$-valued random variables such that* (2) *holds.*
Then, the random variables $U_1, U_2, \ldots, U_r$, defined by equation (1)*, are the order statistics of the vector $(Y_1, \ldots, Y_r)$.*

In what follows we present an interpretation of the marginal distributions of the variables $Y_1, Y_2, \ldots, Y_r$, in terms of the associated occupancy numbers $X_1, X_2, \ldots, X_n$. On this purpose, we now consider a special transformation, that map one occupancy model into another. Two other natural transformations, also intersting for our next developments, will be introduced later on.

2.1 Transformation $\mathcal{K}_1$

The first transformation we consider can be described as follows.
Consider r objects distributed on n cells according to a given occupancy model and let $X_1, X_2, \ldots, X_n$ be the related occupancy numbers. We drop one of the particles from this population randomly (i.e. so that each of the r particles has the same probability to be dropped, independently of its cell). We denote by $X'_1, X'_2, \ldots, X'_n$ the occupancy numbers associated with the new population of $r-1$ particles.
This transformation will be denoted by $\mathcal{K}_1$. Notice that $\mathcal{K}_1 : \mathcal{P}\left(A_{n,r}\right) \longrightarrow \mathcal{P}\left(A_{n,r-1}\right)$.

Lemma 2.1. *Consider the event $E_{\mathbf{x}'} = \{X'_1 = x'_1, \ldots, X'_n = x'_n\}$. Then,*

$$P\left\{E_{\mathbf{x}'}\right\} = \sum_{h=1}^{n} \frac{x'_h + 1}{r} P\{X_1 = x'_1, \ldots, X_h = x'_h + 1, \ldots, X_n = x'_n\}. \tag{6}$$

Proof. The proof consists of simple manipulations. In fact, we can write

$$\begin{aligned}
P\{E_{\mathbf{x}'}\} &= P\{X_1' = x_1', \ldots, X_n' = x_n'\} \\
&= \sum_{h=1}^{n} P\{E_{\mathbf{x}'} \cap (\text{the eliminated particle is from the cell } h)\} \\
&= \sum_{h=1}^{n} P\{E_{\mathbf{x}'} \cap \big(X_1 = x_1', \ldots, X_h = x_h' + 1, \ldots, X_n = x_n'\big)\} \\
&= \sum_{h=1}^{n} P\{E_{\mathbf{x}'} | X_1 = x_1', \ldots, X_h = x_h' + 1, \ldots, X_n = x_n'\} \times \\
&\qquad\qquad \times P\{X_1 = x_1', \ldots, X_h = x_h' + 1, \ldots, X_n = x_n'\} \\
&= \sum_{h=1}^{n} \frac{x_h' + 1}{r} P\{X_1 = x_1', \ldots, X_h = x_h' + 1, \ldots, X_n = x_n'\}\,.
\end{aligned}$$

□

Let us now consider the exchangeable vectors $(Y_1, \ldots, Y_r)$ and $\big(Y_1', \ldots, Y_{r-1}'\big)$, corresponding to the occupancy numbers $X_1, \ldots, X_n$ and $X_1', \ldots, X_n'$, respectively.

Proposition 2.2. *The joint distribution of* $\big(Y_1', \ldots, Y_{r-1}'\big)$ *coincides with the* $(r-1)$*-dimensional marginal distribution of* $(Y_1, \ldots, Y_r)$*.*

Proof. In view of (4) and (6), the joint distribution of $\big(Y_1', \ldots, Y_{r-1}'\big)$ is given by

$$P\{Y_1' = y_1', Y_2' = y_2', \ldots, Y_{r-1}' = y_{r-1}'\} =$$

$$= \frac{P\{X_1' = \tilde{\varphi}_1(\mathbf{y}'), X_2' = \tilde{\varphi}_2(\mathbf{y}'), \ldots, X_n' = \tilde{\varphi}_n(\mathbf{y}')\}}{\binom{r-1}{\tilde{\varphi}_1(\mathbf{y}')\tilde{\varphi}_2(\mathbf{y}')\cdots\tilde{\varphi}_n(\mathbf{y}')}}$$

$$\begin{aligned}
&= \frac{\tilde{\varphi}_1(\mathbf{y}')\tilde{\varphi}_2(\mathbf{y}')\cdots\tilde{\varphi}_n(\mathbf{y}')}{(r-1)!} \sum_{h=1}^{n} \frac{\tilde{\varphi}_h(\mathbf{y}') + 1}{r} \times \\
&\qquad \times P\{X_1 = \tilde{\varphi}_1(\mathbf{y}'), \ldots, X_h = \tilde{\varphi}_h(\mathbf{y}') + 1, \ldots, X_n = \tilde{\varphi}_n(\mathbf{y}')\}.
\end{aligned}$$

For brevity sake, let us set

$$\tilde{\boldsymbol{\varphi}}^{(\boldsymbol{h})}(\mathbf{y}') := \big(\tilde{\varphi}_1(\mathbf{y}'), \ldots, \tilde{\varphi}_h(\mathbf{y}') + 1, \ldots, \tilde{\varphi}_n(\mathbf{y}')\big)\ ,$$

so that we rewrite the previous equation as

$$P\{Y'_1 = y'_1, Y'_2 = y'_2, \ldots, Y'_{r-1} = y'_{r-1}\} = \\ = \frac{\tilde{\varphi}_1(\mathbf{y}')\tilde{\varphi}_2(\mathbf{y}')\cdots\tilde{\varphi}_n(\mathbf{y}')}{(r-1)!}\sum_{h=1}^{n}\frac{\tilde{\varphi}_h(\mathbf{y}')+1}{r}P\{\mathbf{X} = \tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\}.$$

We can now use (5) and get

$$P\{Y'_1 = y'_1, Y'_2 = y'_2, \ldots, Y'_{r-1} = y'_{r-1}\} =$$

$$= \frac{\tilde{\varphi}_1(\mathbf{y}')!\tilde{\varphi}_2(\mathbf{y}')!\cdots\tilde{\varphi}_n(\mathbf{y}')!}{(r-1)!}\sum_{h=1}^{n}\frac{\tilde{\varphi}_h(\mathbf{y}')+1}{r}\frac{r!}{\tilde{\varphi}_1(\mathbf{y}')!\cdots(\tilde{\varphi}_h(\mathbf{y}')+1)!\cdots\tilde{\varphi}_n(\mathbf{y}')!}\times \\ \times P\left\{Y_1 = \psi_1\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right), \ldots, Y_{r-1} = \psi_{r-1}\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right), Y_r = \psi_r\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right)\right\}$$

$$= \sum_{h=1}^{n} P\left\{Y_1 = \psi_1\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right), \ldots, Y_{r-1} = \psi_{r-1}\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right), Y_r = \psi_r\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right)\right\}.$$

Let us focus on the vector

$$\left(\psi_1\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right), \ldots, \psi_{r-1}\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right), \psi_r\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right)\right).$$

The first $r-1$ components are defined by the composition $(\psi \circ \tilde{\varphi})(\mathbf{y}')$ and the result is a $(r-1)$-tuple

$$\left(y'_{\sigma(1)}, \ldots, y'_{\sigma(r-1)}\right) \in B_{r-1,n}$$

for every permutation σ of $\{1, 2, \ldots, r-1\}$. The last component $\psi_r\left(\tilde{\boldsymbol{\varphi}}^{(h)}(\mathbf{y}')\right)$ is not determined by the previous composition. It is a "free" element indicating the cell in which the additional r-th particle fell. Thus, it is a value randomly chosen in $\{1, \ldots, n\}$.
Finally, by recalling the exchangeability of $(Y_1, \ldots, Y_r)$, we get the equality

$$P\{Y'_1 = y'_1, Y'_2 = y'_2, \ldots, Y'_{r-1} = y'_{r-1}\} = \sum_{h=1}^{n} P\{Y_1 = y'_1, \ldots, Y_{r-1} = y'_{r-1}, Y_r = h\}.$$

and then the conclusion follows. □

2.2 Transformation $\mathcal{K}_2$

Here we consider the transformation $\mathcal{K}_2 : \mathcal{P}(A_{n,r}) \longrightarrow \mathcal{P}(A_{n-1,r})$ simply obtained by "erasing" one of the n cells. More precisely, we start once again from an occupancy model in $\mathcal{P}(A_{n,r})$ and let $X_1, \ldots, X_n$ denote occupancy numbers jointly distributed according to such

a model. We now consider the case where the n-th cell is eliminated and any of the X_n particles that had fallen within it, is put at random, and independently, within the remaining cells (see also details about the MB model later on).
By denoting $X_1'', \dots, X_{n-1}''$ the occupancy numbers obtained by this procedure, and for $\left(x_1'', \dots, x_{n-1}''\right) \in A_{n-1,r}$, we can write

$$
\begin{aligned}
&P\{X_1'' = x_1'', \dots, X_{n-1}'' = x_{n-1}''\} = \\
&\qquad = \sum_{x=0}^{r} \sum_{\boldsymbol{\xi} \in A_{n-1,x}} \frac{\binom{x}{\xi_1 \cdots \xi_{n-1}}}{(n-1)^x} P\{X_1 = x_1'' - \xi_1, \dots, X_{n-1} = x_{n-1}'' - \xi_{n-1}, X_n = x\}, \quad (7)
\end{aligned}
$$

where we obviously mean $P\{X_1 = x_1, \dots, X_n = x_n\} = 0$, whenever some of the coordinates $x_1, \dots, x_{n-1}$ turns out to be smaller than 0.

2.3 Transformation $\mathcal{K}_{n,s}^{(N,r)}$

Here and in the rest of the paper we will use the notation $S_N = \sum_{i=1}^{N} X_i$. We consider the transformation that maps $\mathcal{P}\left(A_{N,r}\right)$ onto $\mathcal{P}\left(A_{n,s}\right)$ (with $1 \le n \le N-1, 1 \le s \le r$) that is simply obtained by conditioning on a fixed value s for the partial sum S_n. For variables $X_1, \dots, X_N$ distributed according to a given occupancy model in $\mathcal{P}\left(A_{N,r}\right)$ we consider the model in $A_{n,s}$ defined by

$$
P\{X_1 = x_1, \dots, X_n = x_n | S_n = s\} = \frac{P\{X_1 = x_1, \dots, X_n = x_n\}}{P\{S_n = s\}}
$$

for $(x_1, \dots, x_n) \in A_{n,s}$ and with

$$
\begin{aligned}
&P\{X_1 = x_1, \dots, X_n = x_n\} = \\
&\qquad = \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} P\{X_1 = x_1, \dots, X_n = x_n, X_{n+1} = \eta_1, \dots, X_N = \eta_{N-n}\},
\end{aligned}
$$

$$
P\{S_n = s\} = \sum_{\mathbf{x} \in A_{n,s}} \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} P\{X_1 = x_1, \dots, X_n = x_n, X_{n+1} = \eta_1, \dots, X_n = \eta_{N-n}\}.
$$

3 Exchangeable Occupancy Models

Often relevant occupancy models are such that the random variables $X_1, \dots, X_n$ are exchangeable. The class of the Exchangeable Occupancy Models (EOM) is actually a wide and interesting one. We then devote this Section to analyze several special aspects related with such a condition. We notice, in particular, that the most well-known occupancy models, i.e. *Maxwell-Boltzmann*, *Bose-Einstein* and *Fermi-Dirac*, are exchangeable. They are defined as follows.

- Maxwell-Boltzmann:

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} = \frac{1}{n^r} \frac{r!}{x_1! x_2! \cdots x_n!},$$

  for $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n) \in A_{n,r}$.

- Bose-Einstein:

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} = \frac{1}{\binom{n+r-1}{n-1}}$$

  for $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n) \in A_{n,r}$.

- Fermi-Dirac:

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} = \frac{1}{\binom{n}{r}}$$

  for $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n) \in \widehat{A}_{n,r}$, where $\widehat{A}_{n,r} := \{\mathbf{x} \in A_{n,r} : x_j \in \{0, 1\}\}$.

As mentioned above, we can easily see that any vector $(X_1, \ldots, X_n)$ distributed according to one of these models is an exchangeable random vector.

In the next Section we will introduce a wide class of EOM's that contains these fundamental models. For an account about MB, BE, FD see Feller (1968) and Charalambides (2005). In the following Propositions we state some general properties of EOM's.

Proposition 3.1. *If $X_1, X_2, \ldots, X_n$ are exchangeable, then the univariate marginals of Y_1, $Y_2, \ldots, Y_r$ are uniform on $\{1, 2, \ldots, n\}$.*

Proof. We prove the statement by showing that for any $a, b \in \{1, 2, \ldots, n\}$, $a \neq b$, we have $P\{Y_1 = a\} = P\{Y_1 = b\}$. This would mean that $P\{Y_1 = a\} = \frac{1}{n}$ and hence the univariate marginal distributions of $(Y_1, Y_2, \ldots, Y_r)$ are uniform on $\{1, 2, \ldots, n\}$.
Let $(y_2, \ldots, y_r)$ be a $(r-1)$-tuple in $D_{r-1,n}$ and consider $a \in \{1, 2, \ldots, n\}$. Then

$$P\{Y_1 = a\} = \sum_{(y_2, \ldots, y_r) \in D_{r-1,n}} P\{Y_1 = a, Y_2 = y_2, \ldots, Y_r = y_r\}.$$

By using (4) we obtain that

$$P\{Y_1 = a\} = \sum_{(y_2,\ldots,y_r)\in D_{r-1,n}} \frac{P\{X_1 = \tilde{\varphi}_1(a, y_2, \ldots, y_r), \ldots, X_n = \tilde{\varphi}_n(a, y_2, \ldots, y_r)\}}{\binom{r}{\tilde{\varphi}_1(a,y_2,\ldots,y_r)\tilde{\varphi}_2(a,y_2,\ldots,y_r)\cdots\tilde{\varphi}_n(a,y_2,\ldots,y_r)}}$$

$$= \sum_{(y_2,\ldots,y_r)\in D_{r-1,n}} \frac{\tilde{\varphi}_1(a, y_2, \ldots, y_r)! \cdots \tilde{\varphi}_n(a, y_2, \ldots, y_r)!}{r!} \times$$
$$\times P\{X_1 = \tilde{\varphi}_1(a, y_2, \ldots, y_r), \ldots, X_n = \tilde{\varphi}_n(a, y_2, \ldots, y_r)\}$$

Taking into account that for any $j \in \{1, 2, \ldots, n\}$, the function $\tilde{\varphi}_j$ is given by (3), we obtain

$$P\{Y_1 = a\} = \sum_{(y_2,\ldots,y_r)\in D_{r-1,n}} \frac{\tilde{\varphi}_1(y_2, \ldots, y_r)! \cdots (\tilde{\varphi}_a(y_2, \ldots, y_r) + 1)! \cdots \tilde{\varphi}_n(y_2, \ldots, y_r)!}{r!} \times$$
$$\times P\{X_1 = \tilde{\varphi}_1(y_2, \ldots, y_r), \ldots, X_a = \tilde{\varphi}_a(y_2, \ldots, y_r) + 1, \ldots, X_n = \tilde{\varphi}_n(y_2, \ldots, y_r)\}$$

$$= \sum_{(y_2,\ldots,y_r)\in D_{r-1,n}} \frac{\tilde{\varphi}_1(y_2, \ldots, y_r)! \cdots \tilde{\varphi}_n(y_2, \ldots, y_r)!}{r!} (\tilde{\varphi}_a(y_2, \ldots, y_r) + 1)$$
$$\times P\{X_1 = \tilde{\varphi}_1(y_2, \ldots, y_r), \ldots, X_a = \tilde{\varphi}_a(y_2, \ldots, y_r) + 1, \ldots, X_n = \tilde{\varphi}_n(y_2, \ldots, y_r)\}. \quad (8)$$

Now, we take $b \in \{1, 2, \ldots, n\}$, $b \neq a$. Without loss of generality we can suppose that $a < b$. Then, the univariate marginal of Y_1 computed in b is

$$P\{Y_1 = b\} = \sum_{(y_2,\ldots,y_r)\in D_{r-1,n}} \frac{\tilde{\varphi}_1(y_2, \ldots, y_r)! \cdots \tilde{\varphi}_n(y_2, \ldots, y_r)!}{r!} (\tilde{\varphi}_b(y_2, \ldots, y_r) + 1)$$
$$\times P\{X_1 = \tilde{\varphi}_1(y_2, \ldots, y_r), \ldots, X_b = \tilde{\varphi}_b(y_2, \ldots, y_r) + 1, \ldots, X_n = \tilde{\varphi}_n(y_2, \ldots, y_r)\}. \quad (9)$$

The sums (8) and (9) have n^{r-1} terms and are computed using all possible $(r-1)$-tuples $(y_2, \ldots, y_r) \in D_{r-1,n}$. Therefore for any $j \in \{1, 2, \ldots, n\}$ and $i \in \{0, 1, 2, \ldots, r-1\}$ there are summation terms in (8) and in (9), such that $\tilde{\varphi}_j(y_2, \ldots, y_r) = i$. Let $\alpha, \beta \in \{0, 1, \ldots, r-1\}$ two fixed values. Then there are summation terms in (8) such that $\tilde{\varphi}_a(y_2, \ldots, y_r) = \alpha$ and $\tilde{\varphi}_b(y_2, \ldots, y_r) = \beta$. Also there are summation terms in (9) such that $\tilde{\varphi}_a(y_2, \ldots, y_r) = \beta$ and $\tilde{\varphi}_b(y_2, \ldots, y_r) = \alpha$.
Let us now consider $(x_1, \ldots, x_a, \ldots, x_b, \ldots, x_n) \in A_{n,r-1}$ such that $x_a = \alpha$ and $x_b = \beta$. Then, there are $\binom{r-1}{x_1 x_2 \cdots x_n}$ summation terms in (8), all equal to

$$\frac{x_1! \cdots x_n!}{r!} (\alpha + 1) P\{X_1 = x_1, \ldots, X_a = \alpha + 1, \ldots, X_b = \beta, \ldots, X_n = x_n\}$$

The overall sum of these terms is:

$$\frac{\alpha+1}{r}P\{X_1 = x_1, \dots, X_a = \alpha+1, \dots, X_b = \beta, \dots, X_n = x_n\}. \tag{10}$$

In the same way let us consider $(x_1, \dots, x_a, \dots, x_b, \dots, x_n) \in A_{n,r-1}$ such that $x_a = \beta$ and $x_b = \alpha$. Then, there are $\binom{r-1}{x_1 x_2 \cdots x_n}$ summation terms in (8), all equal to

$$\frac{x_1! \cdots x_n!}{r!}(\alpha+1)P\{X_1 = x_1, \dots, X_a = \beta, \dots, X_b = \alpha+1, \dots, X_n = x_n\}$$

The overall sum of these terms is:

$$\frac{\alpha+1}{r}P\{X_1 = x_1, \dots, X_a = \beta, \dots, X_b = \alpha+1, \dots, X_n = x_n\}. \tag{11}$$

The random vector $(X_1, X_2, \dots, X_n)$ is exchangeable and therefore (10) and (11) are equal. For any $\alpha, \beta \in \{0, 1, \dots, r-1\}$ the above terms uniquely determine the sums (8) and (9), which then are equal. As a consequence, $P\{Y_1 = a\} = P\{Y_1 = b\}$.

$\square$

Remark 3.1. Let

$$\begin{aligned} (n)_r &= n(n+1)\cdots(n+r-1) \\ n^{(r)} &= n(n-1)\cdots(n-r+1) \end{aligned}$$

be the ascending and falling factorial, respectively (see Charalambides (2005)). For MB, BE and FD, the joint distributions of the random variables $Y_1, \dots Y_r$ are given by

- Maxwell-Boltzmann:
$$P\{Y_1 = y_1, \dots, Y_r = y_r\} = \frac{1}{n^r}$$
for $\mathbf{y} \in D_{r,n}$;

- Bose-Einstein:
$$P\{Y_1 = y_1, \dots, Y_r = y_r\} = \frac{\tilde{\varphi}_1(\mathbf{y})! \cdots \tilde{\varphi}_n(\mathbf{y})!}{(n+r-1)(n+r-2)\cdots(n+1)n} = \frac{\prod_{j=1}^n \tilde{\varphi}_j(\mathbf{y})!}{(n)_r}$$
for $\mathbf{y} \in D_{r,n}$;

- Fermi-Dirac:
$$P\{Y_1 = y_1, \dots, Y_r = y_r\} = \frac{\tilde{\varphi}_1(\mathbf{y})! \cdots \tilde{\varphi}_n(\mathbf{y})!}{n(n-1)(n-2)\cdots(n-r+1)} = \frac{\prod_{j=1}^n \tilde{\varphi}_j(\mathbf{y})!}{n^{(r)}}$$

for $\mathbf{y} \in \widehat{D}_{r,n}$, where $\widehat{D}_{r,n} = \Big\{ \mathbf{y} \in D_{r,n} : \sum_{j=1}^{r} y_j \leq n$ and the y_j's are all distinct$\Big\}$.

All these three distributions admit uniform marginals, in agreement with Proposition 3.1. Observe that, in the MB case, the random variables $Y_1, \ldots, Y_r$ are independent and their distribution is uniform over the set $\{1, 2, \ldots, n\}$.

As a remarkable property of the class of the EOM's we have the closure under the transformations $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_{n,s}^{(N,r)}$.

Proposition 3.2. *Let $(X_1, \ldots, X_N)$ be an EOM on $A_{N,r}$ and fix $n < N$. Conditionally on the event $\{S_n = s\}$, the variables $X_1, \cdots, X_n$ are distributed according to an EOM on $A_{n,s}$.*

Proof. It must be proved that conditionally on $\{S_n = s\}$, the vector $(X_1, \ldots, X_n)$ is exchangeable, i.e.

$$P\left\{X_{\sigma(1)} = x_1, \ldots, X_{\sigma(n)} = x_n | S_n = s\right\} = P\left\{X_1 = x_1, \ldots, X_n = x_n | S_n = s\right\}$$

for every permutation σ of $\{1, 2, \ldots, n\}$. It straightforwardly follows from the exchangeability of the random variables $X_1, \ldots, X_N$. Indeed, for any $(x_1, \ldots, x_n) \in A_{n,s}$, we obtain

$$P\Big\{X_{\sigma(1)} = x_1, \ldots, X_{\sigma(n)} = x_n | S_n = s\Big\} =$$

$$= \frac{P\{X_1 = x_{\sigma(1)}, \ldots, X_{\sigma(n)} = x_n\}}{P\{S_n = s\}}$$

$$= \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} \frac{P\{X_1 = x_{\sigma(1)}, \ldots, X_{\sigma(n)} = x_n, X_{n+1} = \eta_1, \ldots, X_N = \eta_{N-n}\}}{P\{S_n = s\}}$$

$$= \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} \frac{P\{X_1 = x_1, \ldots, X_n = x_n, X_{n+1} = \eta_1, \ldots, X_N = \eta_{N-n}\}}{P\{S_n = s\}}$$

$$= \frac{P\{X_1 = x_1, \ldots, X_n = x_n, S_n = s\}}{P\{S_n = s\}}$$

$$= P\{X_1 = x_1, \ldots, X_n = x_n | S_n = s\}$$

and this concludes the proof. □

The following Proposition can be easily derived from Lemma 2.1.

Proposition 3.3. *Let $(X_1, \ldots, X_n)$ be an EOM on $A_{n,r}$ and $(X_1', \ldots, X_n')$ be the occupancy numbers obtained by applying the transformation $\mathcal{K}_1$. Then, $(X_1', \ldots, X_n')$ is an EOM on $A_{n,r-1}$.*

The following Proposition can be easily obtained by taking into account equation (7).

Proposition 3.4. *Let $(X_1, \ldots, X_n)$ be an EOM on $A_{n,r}$ and $(X_1', \ldots, X_n')$ be the occupancy numbers obtained by applying the transformation $\mathcal{K}_2$. Then, $(X_1', \ldots, X_n')$ is an EOM on $A_{n-1,r}$.*

Remark 3.2. The transformation $\mathcal{K}_2$ can be seen as a special case of a more general class of transformations: we can consider the case where the n-th cell is eliminated and the X_n particles that had fallen within it are distributed within the remaining cells according to an exchangeable occupancy model. The class of the EOM's is closed also w.r.t. this class of transformations.

4 Occupancy $\mathcal{M}^{(a)}$-models

In this Section we consider a remarkable sub-class of EOM's, that includes the models MB, FD and BE. Such a class can be introduced as follows: fix a function $a : \{0, 1, \ldots\} \longrightarrow \mathbb{R}_+$ and for $r, n \in \mathbb{N}$, set

$$P\{\mathbf{X} = \mathbf{x}\} = \frac{\prod_{j=1}^{n} a(x_j)}{C_{n,r}^{(a)}} \qquad \text{for } \mathbf{x} \in A_{n,r}\,, \tag{12}$$

where

$$C_{n,r}^{(a)} = \sum_{\boldsymbol{\xi} \in A_{n,r}} \prod_{j=1}^{n} a\left(\xi_j\right) \,.$$

In this case we say that $\mathbf{X} \equiv (X_1, \ldots, X_n)$ is distributed according to the exchangeable occupancy model $\mathcal{M}_{n,r}^{(a)}$.

Notice that the Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac models are respectively obtained by letting

$$\begin{array}{llll} \text{MB:} & a(x) = \dfrac{1}{x!}, & & \text{for } x = 0, 1, 2, \ldots \\ \text{BE:} & a(x) = 1, & & \text{for } x = 0, 1, 2, \ldots \\ \text{FD:} & a(0) = 1, \quad a(1) = 1, & a(x) = 0, & \text{for } x = 2, 3, \ldots \end{array}$$

It is interesting at this stage to point out the existence of $\mathcal{M}_{n,r}^{(a)}$-models different from the three above. An example is provided by the Pseudo-contagious models presented in Charalambides (2005).

Example (Pseudo-contagious occupancy model). *This is the model characterized by the joint probability distribution*

$$P\{X_1 = x_1, \dots, X_n = x_n\} = \frac{\binom{s+x_1-1}{x_1} \cdots \binom{s+x_n-1}{x_n}}{\binom{sn+r-1}{r}}, \qquad \textit{for } \mathbf{x} \in A_{n,r}\,.$$

This model belongs to the class $\mathcal{M}^{(a)}_{n,r}$. In fact, it corresponds to the choice of $a(x) = \binom{s+x-1}{x}$.

In a $\mathcal{M}^{(a)}_{n,r}$-model, the joint distribution of the $Y_1, \dots, Y_r$ random variables introduced Section 2 is

$$P\{Y_1 = y_1, Y_2 = y_2, \dots, Y_r = y_r\} = \frac{\prod_{l=1}^{n} a(\tilde{\varphi}_l(\mathbf{y}))\tilde{\varphi}_l(\mathbf{y})!}{r! C^{(a)}_{n,r}}.$$

An approach for constructing $\mathcal{M}^{(a)}_{n,r}$-models is described in Charalambides (2005, Chapter 4) and makes use of i.i.d. random variables. It works as follows. Let $Z_1, \dots, Z_n$ be $\mathbb{N}$-valued i.i.d. random variables with common law

$$P\{Z_i = x\} = q(x) \qquad \text{for } i = 1, \dots, n.$$

and let $S_n = Z_1 + \cdots + Z_n$.
Consider the random vector $(X_1, \dots, X_n)$ with values in $A_{n,r}$ and such that

$$P\{X_1 = x_1, \dots, X_n = x_n\} := P\{Z_1 = x_1, \dots, Z_n = x_n | S_n = r\}.$$

This means that, for any $(x_1, \dots, x_n) \in A_{n,r}$, it holds

$$\begin{aligned} P\{X_1 = x_1, \dots, X_n = x_n\} &= \frac{P\{Z_1 = x_1, \dots, Z_n = x_n\}}{P\{S_n = r\}} \\ &= \frac{\prod_{j=1}^{n} q(x_j)}{P\{S_n = r\}}\,. \end{aligned} \tag{13}$$

The Formula (12) can be seen as a slight generalization of Formula (13); actually $a(x)$ is not necessarily a probability distribution. We can furthermore extend as follows the construction of the approach based on i.i.d. variables. Let $Z_1, \dots, Z_n$ be $\mathbb{N}$-valued conditionally i.i.d. random variables with joint discrete density

$$P\{Z_1 = z_1, \dots, Z_n = z_n\} = \prod_{j=1}^{n} a(z_j) \int_{-\infty}^{+\infty} \exp\left\{ -\theta \sum_{j=1}^{n} z_j \right\} \Lambda(d\theta)\,, \tag{14}$$

Λ being a probability distribution on the real line. Put $S_n = \sum_{j=1}^{n} Z_j$ and let us repeat the same construction as before: consider the random vector $(X_1, \dots, X_n)$ with values in $A_{n,r}$

and such that, for $\mathbf{x} \in A_{n,r}$,

$$P\{X_1 = x_1, \ldots, X_n = x_n\} := P\{Z_1 = x_1, \ldots, Z_n = x_n | S_n = r\}.$$

Then, as it is easily seen, we reobtain the occupancy model in (12). Indeed, we have

$$P\{Z_1 = x_1, \ldots, Z_n = x_n | S_n = r\} = \frac{P\{Z_1 = x_1, \ldots, Z_n = x_n\}}{P\{S_n = r\}}$$

$$= \frac{\prod_{j=1}^{n} a\left(x_j\right) \int_{-\infty}^{+\infty} \exp\{-\theta r\} \Lambda\left(d\theta\right)}{\sum_{\mathbf{v}\in A_{n,r}} \prod_{j=1}^{n} a\left(v_j\right) \int_{-\infty}^{+\infty} \exp\{-\theta r\} \Lambda\left(d\theta\right)}$$

$$= \frac{\prod_{j=1}^{n} a\left(x_j\right)}{C_{n,r}^{(a)}}.$$

Remark 4.1. The fact that $P\{Z_1 = x_1, \ldots, Z_n = x_n | S_n = r\}$ does not depend on the distribution Λ, has an immediate interpretation in statistical terms: for the *exponential model* in (14), S_n is a *sufficient statistic* with respect to the *parameter* θ.

The construction considered above is somehow more general than the one based on i.i.d. variables. Anyway, the Formula (13) suggests some important remarks.

Remark 4.2. Up to a multiplicative factor, the normalization constant $C_{n,r}^{(a)}$ in (12) can be interpreted as the probability that a sum of certain i.i.d. random variables is equal to r, i.e. $C_{n,r}^{(a)} = kP\{S_n = r\}$ with k suitable constant.

Remark 4.3. With the above construction the three fundamental models can be recovered by choosing suitably the distribution $q(x)$. In fact, MB, BE and FD correspond to the cases when $q(x)$ obeys a Poisson, Geometric and Bernoulli distribution, respectively. Moreover, the pseudo-contagious occupancy model is obtained by setting $q(x)$ a negative Binomial distribution.

From Proposition 3.2 we know that a subvector $(X_1, \ldots, X_n)$ of an EOM $(X_1, \ldots, X_N)$ is still an EOM conditionally on a fixed value for the sum S_n (closure property w.r.t. transformations of the type $\mathcal{K}_{n,s}^{(N,r)}$). Then we know that, by applying this transformation to an occupancy model of the form $\mathcal{M}^{(a)}$, we surely obtain an exchangeable model. It is then natural to wonder whether the latter is still of the form $\mathcal{M}^{(a)}$ (i.e., whether the closure property also holds for the class $\mathcal{M}^{(a)}$). The following proposition answers this question.

Proposition 4.1. *Let $(X_1, \ldots, X_N)$ be a $\mathcal{M}_{N,r}^{(a)}$-model and define $S_n = X_1 + \cdots + X_n$ with $n \leq N$. Conditionally on the event $\{S_n = s\}$, the variables $X_1, \cdots, X_n$ are distributed as a $\mathcal{M}_{n,s}^{(a)}$-model.*

Proof. We need to prove that conditionally on $\{S_n = s\}$, the vector $(X_1, \dots, X_n)$ is distributed as a $\mathcal{M}^{(a)}_{n,s}$-model, i.e.

$$P\{X_1 = x_1, \dots, X_n = x_n | S_n = s\} = \frac{\prod_{j=1}^n a(x_j)}{C^{(a)}_{n,s}}.$$

For any $(x_1, \dots, x_n) \in A_{n,s}$, we have

$$\begin{aligned}
P\{X_1 = x_1, \dots, X_n = x_n | S_n = s\} = \\
&= \frac{P\{X_1 = x_1, \dots, X_n = x_n\}}{P\{S_n = s\}} \\
&= \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} \frac{P\{X_1 = x_1, \dots, X_n = x_n, X_{n+1} = \eta_1, \dots, X_N = \eta_{N-n}\}}{P\{S_n = s\}} \\
&= \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} \frac{\prod_{j=1}^n a(x_j) \prod_{i=1}^{N-n} a(\eta_i)}{C^{(a)}_{N,r} \, P\{S_n = s\}} \\
&= K \prod_{j=1}^n a(x_j)
\end{aligned}$$

where

$$K := \frac{1}{C^{(a)}_{N,r} \, P\{S_n = s\}} \sum_{\boldsymbol{\eta} \in A_{N-n,r-s}} \prod_{i=1}^{N-n} a(\eta_i) \, .$$

Since $P\{X_1 = x_1, \dots, X_n = x_n | S_n = s\}$ is a probability distribution, K is exactly equal to $\left(C^{(a)}_{n,s}\right)^{-1}$ and this concludes the proof. □

As seen in Proposition 3.3, the action of dropping one particle at random (transformation $\mathcal{K}_1$) preserves the exchangeability property. As we will see in next Proposition, the structure of $\mathcal{M}^{(a)}$-model is preserved only under the technical assumption (15). It is easy to see that MB, BE, FD and the pseudo-contagious occupancy models satisfy this condition.

Proposition 4.2. *Let $(X_1, X_2, \dots, X_n)$ be a $\mathcal{M}^{(a)}_{n,r}$-model and let the function $a : \{0, 1, \dots\} \to \mathbb{R}_+$ satisfies the condition*

$$\frac{C^{(a)}_{n,r-1}}{C^{(a)}_{n,r}} \sum_{h=1}^n \frac{x'_h + 1}{r} \frac{a(x'_h + 1)}{a(x'_h)} = 1 \, , \qquad \textit{for any } \mathbf{x}' \in A_{n,r-1} \, . \tag{15}$$

Then, the occupancy vector $(X'_1, \dots, X'_n)$, obtained by applying the transformation $\mathcal{K}_1$, is distributed according to the model $\mathcal{M}^{(a)}_{n,r-1}$.

Proof. We have to show that

$$P\{X'_1 = x'_1, \dots, X'_n = x'_n\} = \frac{\prod_{j=1}^n a(x'_j)}{C^{(a)}_{n,r-1}}.$$

In view of (6) the joint distribution of $(X'_1, \dots, X'_n)$ is given by

$$\begin{aligned} P\{X'_1 = x'_1, \dots, X'_n = x'_n\} &= \sum_{h=1}^n \frac{x'_h + 1}{r} P\{X_1 = x'_1, \dots, X_h = x'_h + 1, \dots, X_n = x'_n\} \\ &= \sum_{h=1}^n \frac{x'_h + 1}{r} \frac{a(x'_h + 1)}{C^{(a)}_{n,r}} \prod_{\substack{i=1 \\ i \neq h}}^n a(x'_i) \\ &= \frac{\prod_{j=1}^n a(x'_j)}{C^{(a)}_{n,r-1}} \frac{C^{(a)}_{n,r-1}}{C^{(a)}_{n,r}} \sum_{h=1}^n \frac{x'_h + 1}{r} \frac{a(x'_h + 1)}{a(x'_h)} \\ &= \frac{\prod_{j=1}^n a(x'_j)}{C^{(a)}_{n,r-1}}, \end{aligned}$$

where the last equality follows from (15). □

Remark 4.4. One can easily realize that the class of the $\mathcal{M}^{(a)}$-models is strictly contained within the class of the EOM's. Examples can be easily found, for instance, by starting from models in the class $\mathcal{M}^{(a)}$ and by applying the transformations $\mathcal{K}_1$ or $\mathcal{K}_2$.

5 Processes with the generalized UOSP

For $M \in \mathbb{N} \cup \{+\infty\}$, let $\{N_t\}_{t=0,1,\dots,M}$ be a discrete-time counting process with *jump amounts* $J_0, J_1, \dots, J_M$, i.e. $J_0, J_1, \dots, J_M$ is a sequence of $\{0, 1, 2, \dots\}$-valued random variables and, for $t = 0, 1, \dots, M$,

$$N_t = \sum_{h=0}^t J_h. \tag{16}$$

For our purposes we introduce the following notation and definitions.
Let $T_1, T_2, \dots$ denote the *arrival times* of the process $\{N_t\}_{t=0,1,\dots}$, i.e.

$$T_n = \inf\{t \geq 0 : N_t \geq n\}, \tag{17}$$

and let $Z_1, Z_2, \dots$ denote the *inter-arrival times*, i.e.

$$Z_n = T_n - T_{n-1}\,.$$

Notice that, since $P\{J_h > 1\} > 0$, it can happen $\{T_{n-1} = T_n\}$, and then $\{Z_n = 0\}$, for some n. The zeros of the Z's are related to the *ties* of the T's and to the jump amounts greater than one.

Definition 5.1. Let $\{N_t\}_{t=0,1,\ldots}$ be a discrete-time counting process with jump amounts $J_0, \ldots, J_t$. We say that it satisfies the $\mathcal{M}^{(a)}$-Uniform Order Statistics Property ($\mathcal{M}^{(a)}$-UOSP) if, for any $t, k \in \mathbb{N}$ and any $(j_0, \ldots, j_t) \in A_{t+1,k}$, we have

$$P\{J_0 = j_0, \ldots, J_t = j_t | N_t = k\} = \frac{\prod_{h=0}^{t} a(j_h)}{C^{(a)}_{t+1,k}}. \tag{18}$$

This definition can be seen as a natural, and unifying extension of the definition of discrete UOSP given in Shaked, Spizzichino and Suter (2004, 2008).

Definition 5.2. Let $a : \{0, 1, \ldots\} \longrightarrow \mathbb{R}_+$ and $R_t : \{0, 1, \ldots\} \longrightarrow \mathbb{R}_+$ be given functions. The process $\{N_t\}_{t=0,1,\ldots,M}$ is an *a-mixed-geometric process* if the discrete joint density of $(J_0, J_1, \ldots, J_t)$ has the form

$$p_t(j_0, j_1, \ldots, j_t) = R_t\left(\sum_{h=0}^{t} j_h\right) \cdot \prod_{h=0}^{t} a(j_h) \quad \text{for } t = 0, 1, \ldots, M. \tag{19}$$

Remark 5.1. The sequence of functions $R_1, R_2, \ldots$ in the Definition 5.2 cannot be independent of function a. In fact, since the discrete density p_{t-1} must be the marginal of p_t, from (19) we obtain

$$R_{t-1}(k) = \sum_{l=0}^{+\infty} a(l) R_t(k + l).$$

We can now formulate the following Theorem, that extends the results in Shaked, Spizzichino and Suter (2004, Section 4).

Theorem 5.1. *Let $\{N_t\}_{t=0,1,\ldots}$ be a discrete-time counting process. Then, the following statements are equivalent:*

(i) $\{N_t\}_{t=0,1,\ldots}$ *satisfies the* $\mathcal{M}^{(a)}$*-UOSP.*

(ii) $\{N_t\}_{t=0,1,\ldots}$ *is an a-mixed-geometric process.*

(iii) For any $k \in \mathbb{N}$, the discrete joint density of the inter-arrival times $Z_1, Z_2, \ldots, Z_k$ is given by

$$P\{Z_1 = z_1, \ldots, Z_k = z_k\} = R_{\sum_{i=1}^{k} z_i}(k) \cdot \prod_{h=0}^{\sum_{i=1}^{k} z_i} a\left(\sum_i \mathbf{1}_{\{\sum_{d=1}^{i} z_d = h\}}\right). \tag{20}$$

(iv) For any $k \in \mathbb{N}$, the discrete joint density of the arrival times $T_1, T_2, \ldots, T_k$ is given by

$$P\{T_1 = t_1, \ldots, T_k = t_k\} = R_{t_k}(k) \cdot \prod_{h=0}^{t_k} a\left(\sum_i \mathbf{1}_{\{T_i = h\}}\right). \tag{21}$$

Proof. (i) $\Rightarrow$ (ii). Consider $\{N_t\}_{t=0,1\ldots,}$ satisfying the $\mathcal{M}^{(a)}$-UOSP. For any $k \in \mathbb{N}$, we set

$$R_t(k) := \frac{P\{N_t = k\}}{C^{(a)}_{t+1,k}}.$$

Clearly (19) is satisfied. In fact, it easily yields

$$\begin{aligned} P\{J_0 = j_0, \ldots, J_t = j_t\} &= P\left\{J_0 = j_0, \ldots, J_{t-1} = j_{t-1}, N_t = \sum_{h=0}^{t} j_h\right\} \\ &= P\left\{J_0 = j_0, \ldots, J_{t-1} = j_{t-1} \,\middle|\, N_t = \sum_{h=0}^{t} j_h\right\} P\left\{N_t = \sum_{h=0}^{t} j_h\right\} \\ &= \frac{\prod_{h=0}^{t} a(j_h)}{C^{(a)}_{t+1,\sum_{h=0}^{t} j_h}} P\left\{N_t = \sum_{h=0}^{t} j_h\right\}, \end{aligned}$$

where the last equality is due to (18).

(ii) $\Rightarrow$ (i). Consider $\{N_t\}_{t=0,1\ldots,}$ an a-mixed-geometric process. For any $t, k \in \mathbb{N}$ and for any $(j_0, \ldots, j_t) \in A_{t+1,k}$, by (19) we get

$$\begin{aligned} P\{J_0 = j_0, \ldots, J_t = j_t | N_t = k\} &= \frac{P\{J_0 = j_0, \ldots, J_t = j_t\}}{P\{N_t = k\}} \\ &= \frac{R_t(k) \cdot \prod_{h=0}^{t} a(j_h)}{\sum_{\mathbf{v} \in A_{t+1,k}} R_t(k) \cdot \prod_{h=0}^{t} a(v_h)} = \frac{\prod_{h=0}^{t} a(j_h)}{C^{(a)}_{t+1,k}}, \end{aligned}$$

which is exactly (18).

(ii) $\Rightarrow$ (iii). Consider $\{N_t\}_{t=0,1\ldots,}$ an a-mixed-geometric process and for any $k \in \mathbb{N}$, let $z_1, \ldots, z_k$ be positive integers. We have

$$P\{Z_1 = z_1, \ldots, Z_k = z_k\} = P\left\{T_1 = z_1, T_2 = z_1 + z_2, \ldots, T_k = \sum_{d=1}^{k} z_d\right\}.$$

Moreover, by the definitions of counting process (16) and of arrival times (17), we obtain that the jump amounts are given by

$$J_h = \sum_i \mathbf{1}_{\{T_i = h\}}.$$

Hence, we get

$$
\begin{aligned}
&P\left\{T_1 = z_1, T_2 = z_1 + z_2, \dots, T_k = \sum_{d=1}^{k} z_d\right\} = \\
&\quad = P\left\{J_0 = \sum_i \mathbf{1}_{\{T_i=0\}}, J_{z_1} = \sum_i \mathbf{1}_{\{T_1=z_1\}}, \dots, J_{\sum_{d=1}^k z_d} = \sum_i \mathbf{1}_{\left\{T_i=\sum_{d=1}^k z_d\right\}}\right\} \\
&\quad = R_{\sum_{d=1}^k z_d}\left(J_0 + J_{z_1} + \dots + J_{\sum_{d=1}^k z_d}\right) \prod_{h=0}^{\sum_{d=1}^k z_d} a\left(\sum_i \mathbf{1}_{\{T_i=h\}}\right) \\
&\quad = R_{\sum_{d=1}^k z_d}(k) \prod_{h=0}^{\sum_{d=1}^k z_d} a\left(\sum_i \mathbf{1}_{\left\{\sum_{d=1}^i z_d=h\right\}}\right)
\end{aligned}
$$

and the implication is proved.

(iii) $\Rightarrow$ (ii). Let $L = \sum_{i=0}^t \mathbf{1}_{\{J_i>0\}}$, i.e. the number of J's that are strictly positive. Define

$$
\begin{aligned}
&I_1 := \inf\{0 \le i \le t | J_i > 0\} \\
&I_n := \inf\{I_{n-1} \le i \le t | J_i > 0\}\,, \qquad 1 < n \le L
\end{aligned}
$$

and let

$$\tilde{J}_n := \sum_{h=1}^{n} J_{I_h} \qquad n = 1, 2, \dots, L.$$

Note that the sequence $I_1, \dots, I_L$ is strictly increasing and moreover,

$$\tilde{J}_L = \sum_{h=1}^{L} J_{I_h} = \sum_{h=0}^{t} J_h\,.$$

Without loss of generality, suppose that $\{J_t > 0\}$. Hence $I_L = t$ and we have

$$
\begin{array}{llllllll}
Z_1 & = I_1 & Z_2 & = 0 & \cdots & Z_{\tilde{J}_1} & = 0 \\
Z_{\tilde{J}_1+1} & = I_2 - I_1 & Z_{\tilde{J}_1+2} & = 0 & \cdots & Z_{\tilde{J}_2} & = 0 \\
 & \vdots & & \vdots & & & \vdots \\
Z_{\tilde{J}_{L-1}+1} & = t - I_{L-1} & Z_{\tilde{J}_{L-1}+2} & = 0 & \cdots & Z_{\tilde{J}_L} & = 0.
\end{array}
\tag{22}
$$

From (22), the joint distribution of $(J_0, J_1, \dots, J_t)$ can be written as

$$
\begin{aligned}
P\{J_0 = j_0, J_1 = j_1, \dots, J_t = j_t\} = P\{&Z_1 = i_1, Z_2 = \dots = Z_{\tilde{J}_1} = 0, Z_{\tilde{J}_1+1} = i_2 - i_1, \dots, \\
&Z_{\tilde{J}_{L-1}+1} = t - i_{L-1}, Z_{\tilde{J}_{L-1}+2} = \dots = Z_{\tilde{J}_L} = 0\}\,.
\end{aligned}
$$

Since

$$\sum_{i=1}^{\tilde{j}_l} z_i = t \quad \text{and} \quad \sum_i \mathbf{1}_{\left\{\sum_{d=1}^{i} z_d = h\right\}} = j_h \,,$$

from (20) it follows

$$P\{J_0 = j_0, J_1 = j_1, \ldots, J_t = j_t\} = R_t\left(\sum_{h=0}^{t} j_h\right)\prod_{h=0}^{t} a\left(j_h\right) \,.$$

(iii) $\Rightarrow$ (iv). For any $k \in \mathbb{N}$, consider $t_1, \ldots, t_k$ positive integers. We can write

$$\begin{aligned} P\{T_1 = t_1, \ldots, T_k = t_k\} &= P\{Z_1 = t_1, Z_2 = t_2 - t_1, \ldots, Z_k = t_k - t_{k-1}\} \\ &= R_{\sum_{d=1}^{k} t_d - t_{d-1}}(k) \prod_{h=0}^{\sum_{d=1}^{k} t_d - t_{d-1}} a\left(\sum_i \mathbf{1}_{\left\{\sum_{d=1}^{i} t_d - t_{d-1} = h\right\}}\right) \\ &= R_{t_k}(k) \prod_{h=0}^{t_k} a\left(\sum_i \mathbf{1}_{\{T_i = h\}}\right) \,. \end{aligned}$$

(iv) $\Rightarrow$ (iii). For any $k \in \mathbb{N}$, consider $z_1, \ldots, z_k$ positive integers. We have

$$P\{Z_1 = z_1, \ldots, Z_k = z_k\} = P\left\{T_1 = z_1, T_2 = z_1 + z_2, \ldots, T_k = \sum_{d=1}^{k} z_d\right\}$$

and the conclusion immediately follows by (21).

□

Here, we conclude the paper with a few comments about the processes with the $\mathcal{M}^{(a)}$-UOSP. As it is clear from the definition, the role of such a property is played around conditioning w.r.t. events of the form $\{N_t = k\}$.
Let us consider, in the beginning, a general sequence of exchangeable jump amounts $J_1, J_2, \ldots$ and put $N_s = \sum_{i=1}^{s} J_i$, $s = 1, 2, \ldots$. Then, conditionally on $\{N_t = k\}$, the joint distribution of $J_1, J_2, \ldots, J_t$ is an EOM over $A_{t,k}$. Let $Y_1^{(t)}, \ldots, Y_k^{(t)}$ be the $\{1, 2, \ldots, t\}$- valued, exchangeable, random variables that correspond to such an occupancy model. We denote by $\mathcal{L}^{(t,k)}$ their joint probability law and recall that their common marginal distribution is, in any case, uniform over $\{1, 2, \ldots, t\}$.
In the special case of processes $\{N_s\}_{s=1,2,\ldots}$ with the $\mathcal{M}^{(a)}$-UOSP, $\mathcal{L}^{(t,k)}$ is given by

$$P\left\{Y_1^{(t)} = y_1, Y_2^{(t)} = y_2, \ldots, Y_k^{(t)} = y_k\right\} = \frac{\prod_{h=0}^{t} a(\tilde{\varphi}_h(\mathbf{y}))\tilde{\varphi}_h(\mathbf{y})!}{k!\, C_{t+1,k}^{(a)}} \,.$$

When more in particular $a(x) = 1/x!$, i.e. when the conditional distribution of $J_1, J_2, \ldots, J_t$ is the MB model over $A_{t,k}$, $Y_1^{(t)}, \ldots, Y_k^{(t)}$ are i.i.d. with uniform distribution over $\{1, 2, \ldots, t\}$. This case, the one denoted by UOSP($\leq$) in Shaked, Spizzichino and Suter (2004), can be seen in a sense as the discrete-time analog of the homogeneous Poisson process (in continuous time). The latter satisfies in fact the classical *Order Statistics Property*: conditionally on $\{N_t = k\}$, the arrival times $T_1, \ldots, T_k$ can be seen as the order statistics of k independent random variables, with an uniform distribution over the interval $[0, t]$. The set of the arrivals can thus be seen as a random sample from a uniform distribution over $[0, t]$. In the case of UOSP($\leq$) then we have that, for any $k > 1$ and $t > 1$, $\mathcal{L}^{(t,k-1)}$ obviously coincides with the $(k-1)$-dimensional marginal of $\mathcal{L}^{(t,k)}$. This last circumstance actually is a weaker condition than UOSP($\leq$), in fact it holds for any $\mathcal{M}^{(a)}$-UOSP process with a satisfying the condition (15). It can however be still considered as a condition of randomness for the arrivals, in view of the meaning of the transformation $\mathcal{K}_1$ and of Proposition 2.2.

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